6.1 Vectors in the Plane

Objectives: Students will be able to apply the arithmetic of vectors and use vectors to solve real-world problems.

Some quantities, like temperature, distance, height, area and volume, can be represented by a single real number that indicates magnitude or size. Other quantities, such as force, velocity and acceleration, have magnitude and direction.

While the pair \((a,b)\) determines a point in the plane, it also determines a directed line segment (or arrow) with its tail at the origin and its head at \((a,b)\). Then length of this arrow represents magnitude while the direction in which it points represents direction. Because in this context the ordered pair \((a,b)\) represents a mathematical object with both magnitude and direction, we call it the position vector of \((a,b)\) and denote it as \(<a,b>\) to distinguish it from the point \((a,b)\).

A two-dimensional vector is an ordered pair of real numbers, denoted in component form as \(<a,b>\). The numbers \(a\) and \(b\) are the components of the vector \(v\). The standard representation of the vector \(<a,b>\) is the arrow from the origin to the point \((a,b)\). The magnitude of \(v\) is the length of the arrow, and the direction of \(v\) is the direction in which the arrow is pointing. The vector \(0 = <0,0>\), called the zero vector, has zero length and no direction.
Any two arrows with the same length and pointing in the same direction represent the same vector.

**Head Minus Tail (HMT) Rule** If an arrow has initial point \((x_1,y_1)\) and terminal point \((x_2,y_2)\), it represents the vector \(<x_2 - x_1, y_2 - y_1>\).

**Example** Show that the arrow from \(R = (-4,2)\) to \(S = (-1,6)\) is equivalent to the arrow from \(P = (2,-1)\) to \(Q = (5,3)\).

\[
\overrightarrow{RS} = <-(-4), 6-2> = <3,4> \\
\overrightarrow{PQ} = <5-2, 3-(-1)> = <3,4>
\]

**Magnitude** If \(v\) is represented by the arrow from \((x_1,y_1)\) to \((x_2,y_2)\), then:

\[|v| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.\]

If \(v = <a,b>\), then \[|v| = \sqrt{a^2 + b^2}.\]

**Example** Find the component form and the magnitude of the vector \(v\) represented by \(\overrightarrow{PQ}\), where \(P = (-3, 4)\) and \(Q = (-5, 2)\).

\[
\overrightarrow{PQ} = <-5-(-3), 2-4> = <-2, -2> \text{ component form} \\
|\overrightarrow{v}| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} \text{ magnitude}
\]
In the context of vectors, we refer to numbers as **scalars**.

**Vector Addition and Scalar Multiplication**

Let \( u = <u_1, u_2> \) and \( v = <v_1, v_2> \) and let \( k \) be a real number (scalar).

The sum (or resultant) of the vectors \( u \) and \( v \) is the vector:
\[
 u + v = <u_1 + v_1, u_2 + v_2>.
\]

The sum of the vectors \( u \) and \( v \) can be represented geometrically in two ways:

- **Tail-to-Head**

- **Parallelogram**

The product of the scalar \( k \) and the vector \( u \) is:
\[
k u = k<u_1, u_2> = <ku_1, ku_2>.
\]

**Geometric examples:**
Example  Let $u = \langle -1,3 \rangle$ and $v = \langle 4,7 \rangle$. Find the component form of the following vectors.

a.) $2u = \langle -2,6 \rangle$

b.) $u - v = \langle -5,-4 \rangle$

c.) $2u + 3v = \langle -2,6 \rangle + \langle 12,21 \rangle = \langle 10,27 \rangle$

d.) $3u + (-1)v = \langle -3,9 \rangle + \langle -4,-7 \rangle = \langle -7,2 \rangle$

Unit Vectors: A vector $u$ with length $|u| = 1$.

If $v$ is not the zero vector $\langle 0,0 \rangle$, then the vector $\frac{v}{|v|}$ is a unit vector in the direction of $v$.

Unit vectors provide a way to represent the direction of any nonzero vector. Any vector in the direction of $v$, or the opposite direction, is a scalar multiple of this unit vector $u$.

Example Find a unit vector in the direction of $v = \langle -3,2 \rangle$.

$$|v| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

$$\hat{u} = \frac{\langle -3,2 \rangle}{\sqrt{13}} = \langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$$
The two unit vectors $i = <1,0>$ and $j = <0,1>$ are the standard unit vectors.

Any vector $v$ can be written as an expression in terms of unit vectors. Let's show this.

Let $v = <a,b>$.

$$v = <a,b> = a<i> + b<j>\]

$$v = ai + bj$$

**Example** Find the unit vector in the direction of $u = <3,-4>$.

Write the answer in component form and as a linear combination of $i$ and $j$.

$$|u| = \sqrt{3^2 + (-4)^2} = 5$$

$$v = \text{unit vector} = \left< \frac{3}{5}, -\frac{4}{5} \right> = \frac{3}{5}i - \frac{4}{5}j$$

If $v$ has direction angle $\theta$, the components of $v$ can be computed using the formula $v = <|v|\cos\theta, |v|\sin\theta>$.

From the formula above, it follows that the unit vector in the direction of $v$ is:

$$v = |v|<\cos\theta, \sin\theta>$$

**Example** Find the component form of vector $v$.

$$v = <14\cos 55, 14\sin 55>$$

$$v = <8.030, 11.468>$$

$$\sin 55 = \frac{1}{14}$$

$$\cos 55 = \frac{x}{14}$$

$$y = 14\sin 55$$

$$x = 14\cos \theta$$
Example  Find the magnitude and direction angle of $v = <-2,-5>$. 

The _______ of a moving object is a vector because it has both magnitude and direction. The magnitude of velocity is ______.

Examples
1.) A DC-10 jet aircraft is flying on a bearing of $65^\circ$ at 500 mph. Find the component form of the velocity of the airplane. Note: The bearing is the angle that the line of travel makes with due north, measured clockwise.
2.) A pilot’s flight plan is to fly due east. There is a 65 mph wind with the bearing 60°. Find the compass heading the pilot should follow and determine what the airplane’s ground speed will be (assuming that the speed with no wind is 450 mph).

3.) A force of 30 lbs just keeps the box from sliding down the ramp inclined at 20°. Find the weight of the box.
4.) During one part of its migration, a salmon is swimming at 6 mph and the current is flowing downstream at 3 mph at an angle of 7°. How fast is the salmon moving upstream?

Homework: 1-47 odd